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The generalized Wiener process II: Finite systems

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Abstract. A class of Langevin-like equations (non-Markovian processes) are studied in the presence of non-natural boundary conditions. Exact results for all cumulants and the corresponding Kolmogorov hierarchy of distributions are given in terms of our functional approach we previously reported (1997 *J. Phys. A: Math. Gen.* **30** 8427). The generalized Wiener processes—on finite domains—are completely characterized for reflecting and periodic boundary conditions. Some examples are given to show the behaviour of the moments and the probability distributions for different noises. The interplay between the boundary conditions and the structure of the noises is also pointed out.

1. Introduction

The behaviour of systems under the effect of noise has attracted the interest of many workers for many years [1, 2]. In particular, stochastic equations to model relaxation have been studied for several purposes [3, 4], and by means of different approximations [5]. As is well known, when the random force is Gaussian and white, Fokker–Planck equations for the distributions are available. In general, if any other noise is utilized, an individual and particular treatment is required. This is true even with linear stochastic differential equations (SDEs). Nevertheless, general methods to characterize some non-Markovian processes can be constructed [6]. In [6] (from now on referred to as paper I) we developed a functional approach in order to characterize Langevin-like equations—with natural boundary conditions—and driven by arbitrary structures of noise. From that approach it is possible to construct the characteristic functional of the processes, from which all statistical information can be obtained, hence providing a systematic way to calculate exact properties for a large class of non-Markovian processes. We remark that in order to obtain the characteristic functional of the non-Markovian processes, it is only necessary to know the characteristic functional of the noise.

Recently there has been some interest in the effects of the boundaries on finite non-Markovian diffusion systems. This problem, to our knowledge, has only been treated with dichotomous noise [7]. Therefore, the principal object of this paper is to extend our functional approach to a particular class of *finite* non-Markovian processes, i.e. when there exist boundary conditions (BC) on the domain of interest $\mathcal{D}$ and when the noise—in the corresponding SDE—is an arbitrary stochastic process (SP) $\boldsymbol{\xi}(t)$ characterized by its functional $G_{\xi}([k(t)])$. Once again we note that our approach is *exact* and provides the starting point to obtain, in a systematic way, higher-order cumulants and also the whole Kolmogorov hierarchy.

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In this paper we prove two theorems in order to be able to study a family of finite systems with periodic and reflecting BC. We conclude that if we know the characteristic functional of the unbounded process $G_{X_0}([k(t)])$ (i.e. the SP $\boldsymbol{X}_0(t)$ with natural BC) the restricted process $\boldsymbol{X}(t)$ (i.e. the finite system) can be completely characterized. We remark that due to the fact that the SP $\boldsymbol{X}(t)$ could be non-Markovian, there is not a clear partial differential equation available for its 1-time probability distribution, and this fact is even worse if we want to know some n-time joint probability distribution; thus we make not use of any partial differential equation approach. This is why we are going to start constructing, explicitly, the stochastic realizations satisfying the BC using the well known method of images [9].

We have exemplified our method by calculating probability distributions and moments of some generalized Wiener SP. Then we extend previous studies, in several respect, to analyse the interplay between the BC and the structure of different random forces, for example Gaussian, dichotomous, Poisson and radioactive noises. Even when some of these structures of noise induce in the distribution $P(x, t)$ *rare events*, which look like a discontinuity (Dirac delta boundary or front moving in the distribution), we have shown that their corresponding characteristic functions are continuous, leading therefore to continuous average values. As a result of the noise structure, we are going to show that these averages have—in general—an unusual oscillatory behaviour. In the case when $\boldsymbol{\xi}(t)$ is a Gaussian white noise, of course, our results coincide with those that can be obtained by Fokker–Planck dynamics.

The *time-dependent* generalized Wiener SP $\boldsymbol{X}(t)$ (i.e. when its coefficient is explicitly *time-dependent*) can also be obtained in a similar way, as we reported in paper I. This case corresponds to the analysis of a non-autonomous SDE with BC and in presence of an arbitrary random force $\boldsymbol{\xi}(t)$.

A semi-infinite domain has also been worked out. The generalized Ornstein–Uhlenbeck process $\boldsymbol{V}(t)$, which can also be seen as a circuit equation for a (L-R) electric system in the presence of an arbitrary fluctuating emf [8], is a particular case—with an even potential—that can be analysed in the present framework.

In appendix A we show the general expressions for the whole Kolmogorov hierarchy. In order to make this paper self-contained, in appendix B we give a small review of paper I and present some noise functionals. Furthermore, we present the 1-time probability of the generalized Wiener process with natural BC, from which a comparison with the bounded cases is useful.

2. Finite systems

In section 2 we study a generalized Langevin equation with some special BC on $\mathcal{D} \equiv [-L, L]$.

2.1. Periodic boundary condition on $\mathcal{D}$

Definition 1. *Let the SDE be*

$$\frac{\mathrm{d}X_0(t)}{\mathrm{d}t} = U'(X_0) + \xi(t) \qquad X_0(t) \in [-\infty, \infty] \tag{2.1}$$

with $U'(X_0) \equiv \frac{\mathrm{d}}{\mathrm{d}X_0}U(X_0)$, $U(X_0) = U(X_0 + 2L)$ a periodic potential, and $\xi(t)$ an arbitrary noise. The conditional probability distribution associated to (2.1) and satisfying periodic BC on $\mathcal{D}$ can be built up in terms of the method of images [10,11] as:

$$P(x, t) = W(x) \sum_{m=-\infty}^{+\infty} P_0(x + 2mL, t) \tag{2.2}$$

where $W(x)$ is the window function [12]

$$W(x) = \begin{cases} 1 & \text{if} \quad -L < x < L \\ \frac{1}{2} & x = \pm L \\ 0 & \text{if} \quad x > L \quad \text{and} \quad x < -L \end{cases} \tag{2.3}$$

and $P_0(x, t)$ is the corresponding probability distribution of the SDE (2.1) with natural BC.

Theorem 1. *Let $\boldsymbol{X}_0(t)$ be any stochastic realization solution of the unbounded SDE (2.1). The realizations of the window SP*

$$\boldsymbol{X}(t) = \sum_{m=-\infty}^{+\infty} W(\boldsymbol{X}_m(t))\boldsymbol{X}_m \qquad \boldsymbol{X}_m(t) = \boldsymbol{X}_0(t) + 2mL \tag{2.4}$$

satisfy periodic BC on $\mathcal{D}$. Furthermore the conditional probability distribution of $\boldsymbol{X}(t)$ is given by the distribution (2.2).

Proof. It is simple to show that the stochastic realizations $\boldsymbol{X}(t)$ appearing in (2.4) satisfy periodic BC on $\mathcal{D}$. This can be done by the use of a method of dynamical images. The realizations $\boldsymbol{X}_m(t) = \boldsymbol{X}_0(t) + 2mL$ with $\{m = 0, \pm 1, \pm 2, \ldots\}$ represent unbounded realizations with initial conditions (*images*) in the intervals$[L, 3L], [3L, 5L], \ldots$ for $m = 1, 2, \ldots$; and in the intervals $[-3L, -L], [-5L, -3L], \ldots$ for $m = -1, -2, \ldots$. Thus, due to the occurrence of the *window* functions in (2.4), there will be always *only one* stochastic realization into $\mathcal{D}$. We note that to assure that any image has the same potential as the realization $X_0(t)$ in $\mathcal{D}$, it is necessary that the potential $U(X_0)$ be periodic—of periodicity $2L$—otherwise the method of images fails.

In order to prove the second part of the theorem, it is more convenient to first give two propositions.

Proposition 1. *The realizations of the SP $\boldsymbol{X}(t)$ to the power n are:*

$$\boldsymbol{X}^n(t) = \sum_{m=-\infty}^{+\infty} W(\boldsymbol{X}_m(t))\boldsymbol{X}_m^n(t) \qquad n \in \textit{ natural}. \tag{2.5}$$

Proof. We prove this proposition by induction. For the first step, $n = 1$, the proposition is valid. We assume that for an arbitrary n (2.5) is valid and we prove that it is true for $n + 1$:

$$\boldsymbol{X}^{n+1}(t) = \sum_{m=-\infty}^{+\infty} W(\boldsymbol{X}_m(t))\boldsymbol{X}_m^n(t) \cdot \sum_{l=-\infty}^{+\infty} W(\boldsymbol{X}_l(t))\boldsymbol{X}_l(t)$$

then (2.5) will be true for $n + 1$ if $W(\boldsymbol{X}_m(t)) \cdot W(\boldsymbol{X}_l(t)) = \delta_{ml} W(\boldsymbol{X}_m(t))$. It is possible to show that

$$W(\boldsymbol{X}_m(t)) \cdot W(\boldsymbol{X}_l(t)) = 1 \qquad \text{if} \quad \begin{cases} -L < \boldsymbol{X}_0(t) + 2mL < L \\ \text{and} \\ -L < \boldsymbol{X}_0(t) + 2lL < L \end{cases} \tag{2.6}$$

otherwise it is null. So from inequality (2.6) it is simple to see that $-1 < m - l < 1$. Therefore $m = l$ because m and l are integers. □

Proposition 2. *For any fixed time t the characteristic function of the SP $\boldsymbol{X}(t)$ can be written as*

$$G_X(k,t) = \sum_{m=-\infty}^{+\infty} \langle W(\boldsymbol{X}_m(t)) \exp \mathrm{i}k\boldsymbol{X}_m(t)\rangle \tag{2.7}$$

where the average $\langle \ldots \rangle$ represents the average over the ensemble of realizations of the arbitrary noise $\xi(t)$.

Proof. It follows immediately from the fact that Taylor's coefficients of $G_X(k,t)$ are the moments of $\boldsymbol{X}(t)$, i.e.

$$\langle (\boldsymbol{X}(t))^n \rangle = (-\mathrm{i})^n \frac{\mathrm{d}^n}{\mathrm{d}k^n} G_X(k,t)|_{k=0} \tag{2.8}$$

in correspondence with proposition 1. □

Armed with these propositions we can now prove theorem 1. From the definition of the characteristic function $G_X(k,t)$, its 1-time probability distribution is given by Fourier inversion. Then from (2.7) follows

$$P(x,t) = \sum_{m=-\infty}^{+\infty} \langle W(\boldsymbol{X}_m(t))\delta(\boldsymbol{X}_m(t) - x)\rangle. \tag{2.9}$$

By the property of the Dirac delta function $f(x)\delta(x-a) = f(a)\delta(x-a)$, the probability distribution reads

$$P(x,t) = W(x) \sum_{m=-\infty}^{+\infty} \langle \delta(\boldsymbol{X}_m(t) - x)\rangle. \tag{2.10}$$

Finally from van Kampen's lemma [1], for any SP $\boldsymbol{X}(t)$ it is true that $P(x,t) = \langle \delta(\boldsymbol{X}(t) - x)\rangle$, thus (2.2) follows. □

Now, from the next proposition it is possible to characterize completely the finite non-Markovian SP $\boldsymbol{X}(t)$.

Proposition 3. *The characteristic function (2.7) can alternatively be written in the form*

$$G_X(k,t) = \sum_{m=-\infty}^{+\infty} \frac{\sin(m\pi - kL)}{(m\pi - kL)} \left\langle \exp \mathrm{i}\frac{m\pi}{L}\boldsymbol{X}_0(t)\right\rangle. \tag{2.11}$$

Proof. Take the Fourier transform of the *window* function $W(x)$. Then $f(p) = \int_{-\infty}^{\infty} \mathrm{d}x\, W(x)\mathrm{e}^{-\mathrm{i}px} = 2\sin(pL)/p$, thus (2.7) can be slightly rewritten as:

$$G_X(k,t) = \sum_{m=-\infty}^{+\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}p\, f(p) \langle \exp \mathrm{i}(p+k)\boldsymbol{X}_m(t)\rangle. \tag{2.12}$$

Now using that $\boldsymbol{X}_m(t) = \boldsymbol{X}_0(t) + 2mL$, and the Poisson sum [8]:

$$\sum_{m=-\infty}^{+\infty} \exp(\mathrm{i}2\pi m x) = \sum_{m=-\infty}^{+\infty} \delta(x-m) \tag{2.13}$$

equation (2.12) can be written as:

$$\begin{aligned} G_X(k,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}p\, f(p)\langle \exp \mathrm{i}(p+k)\boldsymbol{X}_0(t)\rangle \sum_{m=-\infty}^{+\infty} \exp \mathrm{i}2Lm(p+k) \\ G_X(k,t) &= \sum_{m=-\infty}^{+\infty} \frac{1}{2L} f(p)|_{p=\frac{m\pi}{L}-k} \left\langle \exp \mathrm{i}\frac{m\pi}{L}\boldsymbol{X}_0(t)\right\rangle. \end{aligned} \tag{2.14}$$

Therefore, from the Fourier transform of the *window* function and (2.14), proposition 3 is true. □

Corollary 3.1. *The conditional probability distribution of the SP (2.4) can alternatively be written in the form*

$$P(x,t) = \frac{1}{2L} W(x) \sum_{m=-\infty}^{+\infty} \left\langle \exp \mathrm{i} \frac{m\pi}{L} (\boldsymbol{X}_0(t) - x) \right\rangle. \tag{2.15}$$

Proof. It follows immediately by using the Fourier transform of (2.11). □

Corollary 3.2. *All the 1-time moments and 1-time cumulants of the SP (2.4) can be calculated if we know the characteristic function of the unbounded SP $\boldsymbol{X}_0(t)$.*

Proof. Any 1-time moments of the SP (2.4) can be calculated from (2.11) as

$$\langle (\boldsymbol{X}(t))^n \rangle = \sum_{m=-\infty}^{+\infty} (-\mathrm{i})^n \left. \frac{\mathrm{d}^n}{\mathrm{d}k^n} \frac{\sin(m\pi - kL)}{(m\pi - kL)} \right|_{k=0} \left\langle \exp \mathrm{i} \frac{m\pi}{L} \boldsymbol{X}_0(t) \right\rangle. \tag{2.16}$$

The average quantity—represented in the bracket—is just the characteristic function of unbounded SP $\boldsymbol{X}_0(t)$ evaluated in $\frac{m\pi}{L}$. From (2.16) all the 1-time cumulants can be calculated immediately from the diagrammatic technique introduced in appendix A of paper I. □

Corollary 3.3. *The whole Kolmogorov hierarchy of distributions of the SP $\boldsymbol{X}(t)$ and all n-time moments follows from the characteristic functional of the unbounded SP $\boldsymbol{X}_0(t)$.*

Proof. The proof follows immediately from discrete Fourier transform techniques. For details see appendix A. □

2.2. Applications to the generalized Wiener process: periodic BC

In this section we make some comments concerning the generalized Wiener process $\boldsymbol{X}(t)$ with periodic BC on $\mathcal{D}$. Thus, put $U = 0$ and let $\xi(t)$ be an arbitrary noise in (2.1). Because this system fulfills all the conditions of theorem 1 it applies straightforwardly.

Note that the generalized *time-dependent* Wiener process with periodic BC on $\mathcal{D}$, i.e. characterized by the non-autonomous SDE: $\dot{X}(t) = \gamma(t)\xi(t)$ where $\gamma(t)$ is a sure function of time, can also be calculated in an entirely analogous way, as was commented in paper I (see also section 3 of this paper for an application to the generalized *time-dependent* Orstein–Uhlenbeck process on a semi-infinite domain).

2.2.1. The 1-time probability distribution. Here we exemplify corollary 3.1 for different noises $\xi(t)$ in the SDE $\dot{X}(t) = \xi(t)$. From corollary 3.1, the conditional probability distribution of the SP (2.4) is:

$$P(x,t) = \frac{1}{2L} W(x) \sum_{m=-\infty}^{+\infty} \exp\left(-\mathrm{i}\frac{m\pi}{L}x\right) G_{X_0}\left(\frac{m\pi}{L}, t\right). \tag{2.17}$$

Note that to use this expression we only need to know the 1-time characteristic function of the SP $\boldsymbol{X}_0(t)$ with natural BC. This object is calculated for several noises in appendix B.

(1) Gaussian non-white noise. From (2.17), (B4) and after a little of algebra the 1-time probability distribution of SP $\boldsymbol{X}(t)$ is:

$$P(x,t) = \frac{1}{2L} W(x) \Bigg\{ 1 + 2\sum_{m=1}^{+\infty} \cos\left[\frac{m\pi}{L}(x - x_0)\right] \times \exp\left[-\frac{\Gamma_2}{2}\left(\frac{m\pi}{L}\right)^2 (t + \tau_c(\mathrm{e}^{-t/\tau_c} - 1))\right]\Bigg\}. \tag{2.18}$$

(2) Dichotomous noise. From (2.17) and (B6) it follows that

$$P(x,t) = \frac{1}{2L} W(x) \Bigg\{ 1 + 2\exp(-\lambda t) \sum_{m=1}^{+\infty} \cos\left[\frac{m\pi}{L}(x - x_0)\right] \times \left[\cosh\left(\sqrt{\lambda^2 - (\frac{m\pi}{L}a)^2}t\right) + \frac{\lambda}{\sqrt{\lambda^2 - (\frac{m\pi}{L}a)^2}} \sinh\left(\sqrt{\lambda^2 - (\frac{m\pi}{L}a)^2}t\right)\right]\Bigg\} \tag{2.19}$$

where $x_0 \in \mathcal{D}$ represents the initial condition of the SP with periodic BC.

In figure 1 we have plotted these distributions for two different times ($t = 0.8;\ 3.5$). From these plots we see that at short times and with a dichotomous noise the probability distribution of the SP $\boldsymbol{X}(t)$ spreads no further than the place reached by the Dirac delta contributions (rare events). As with natural BC [13] (see appendix B), the two travelling Dirac-deltas appearing in the probability distribution come from the deterministic realizations of $\xi(t)$, i.e. solutions of $\dot{X}(t) = \pm a$, but now they must satisfy periodic BC on $\mathcal{D}$. Nevertheless, at short times the process has not been able to feel the presence of the boundaries, thus the distribution probability at these times is the same as that in the infinite domain. At short times the probability with Gaussian noise has a faster spreading. This happens because with this noise the SP $\boldsymbol{X}(t)$ is immediately able to explore all the domain $\mathcal{D}$, in contrast with the Dirac delta frontier for the dichotomous case.

At the largest time ($t = 3.5$) and with the dichotomous noise, the Dirac deltas have reached the frontier of $\mathcal{D}$ and therefore they reappear on the opposite side, producing a step in the distribution $P(x,t)$ near the frontier. The similarity of this distribution with the Gaussian one arises because we imposed that both noises had the same amplitudes and the corresponding noise correlation times were similar (i.e. $\tau_c = 0.1$ and $\frac{1}{2\lambda} = 0.5$). The main difference is the Dirac delta contribution. In the asymptotic long-time limit the *probability* weight of the Dirac-deltas is null and therefore the stationary distributions $P(x, t \to \infty)$ are equal. This asymptotic distribution is a constant over the whole domain $\mathcal{D}$.

(3) Poisson noise. From (2.17) and (B9) the 1-time probability distributions of SP $\boldsymbol{X}(t)$ is

$$P(x,t) = \frac{1}{2L} W(x) \sum_{m=-\infty}^{+\infty} \exp\left[-\mathrm{i}\frac{m\pi}{L}(x - x_0) + \rho t(\mathrm{e}^{\mathrm{i}A\frac{m\pi}{L}} - 1)\right]. \tag{2.20}$$

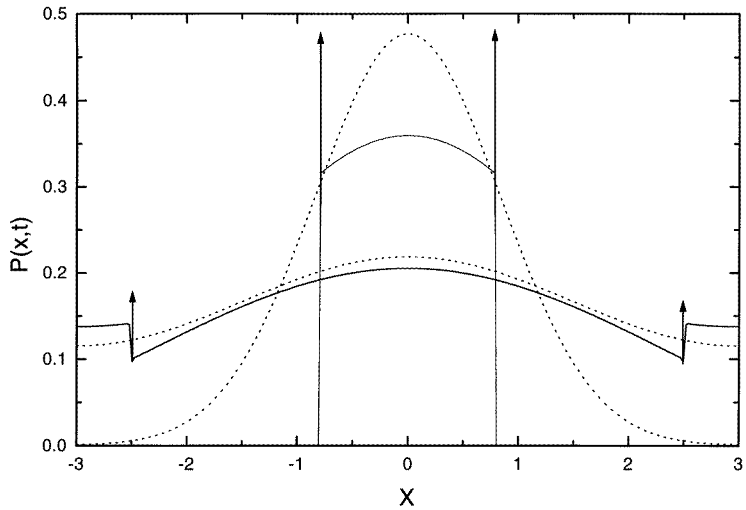

Figure 1. Probability distribution of the SP $\boldsymbol{X}(t)$ for two values of times ($t = 0.8$; 3.5) and for two different structures of noise: Gaussian non-white (dotted curve) and dichotomous (solid curve). The parameters of the Gaussian non-white noise are: $\tau_c = 0.1$; $\Gamma_2 = 1$. The parameters of the dichotomous noise are: $a = 1$; $\lambda = 1$. For all these plots the initial contidion of SP $\boldsymbol{X}(t)$ is $x_0 = 0$, the boundary condition on $[-L, L]$ is periodic, and the length of the domain $[-L, L]$ is $2L = 6$. The arrows represent the location of the rare events (Dirac deltas).

Introducing the Taylor expansion of the exponential function and using the Poisson sum (2.13), results in

$$P(x,t) = W(x)\exp(-\rho t)\sum_{n=0}^{+\infty}\frac{(\rho t)^n}{n!}\sum_{m=-\infty}^{+\infty}\delta(x - x_0 - nA - 2Lm). \qquad (2.21)$$

In comparison with the unbounded case (see appendix B), we now have a more complex selection rule for the allowed values of sites x. To understand which is this set of sites, we must bear in mind that what we have solved is the probability distribution of a particle that at random times does jumps of length $A > 0$ (in the positive direction) and this movement is confined in a torus (ring) of length $2L$. For example, if $x_{0,}$ A and L are natural numbers, the set of sites is restricted to the integer numbers in the interval $[-L, L]$. Therefore, in general, the movement of the particle is not ergodic on $\mathcal{D}$. Only if A or L are irrational numbers, is ergodicity on $\mathcal{D}$ obtained [14].

Suppose now that we have determined the set of allowed sites $\{x\}$. Then from (2.21) it is possible to quantify the probability to be in each site x as a function of time

$$\mathcal{P}_x(t) = \exp(-\rho t)\sum_{n=0}^{+\infty}\frac{(\rho t)^{n_0(x,x_0,A,L)+n\mathcal{T}(A,L)}}{[n_0(x,x_0,A,L)+n\mathcal{T}(A,L)]!} \qquad (2.22)$$

where $\mathcal{P}_x(t)$ is a probability (not a distribution) and it is understood that x is restricted to those allowed sites in $[-L, L]$. The function $n_0(x, x_0, A, L)$ represents the numbers of steps (of length A) needed to reach x for the first time starting from x_0 and 'walking' in the positive direction. Furthermore, $\mathcal{T}(A, L)$ is the 'time of recurrence' for a directed random walk in a torus. This means that $\mathcal{T}(A, L)$ quantifies the number of steps needed to return to an allowed site x, starting from x. Therefore a heuristic interpretation of expression (2.22) is obvious. This equation represents a sum of different paths. The term $n = 0$ gives the probability of arriving at x at time t starting in x_0. In general, the nth term weights the probability when the

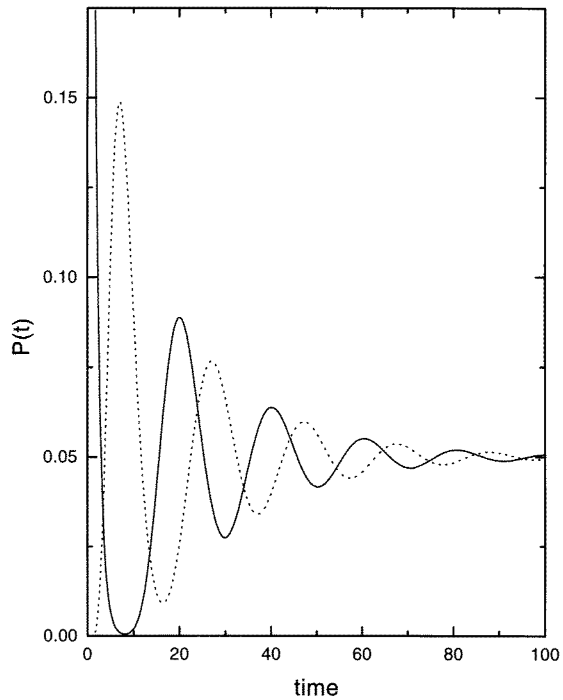

Figure 2. Probability to be at some allowed site of the set $[x]$, in the ring, as a function of time. The solid curve corresponds to the value $x = 0$ and the dotted curve to the value $x = 7$. The parameters of the Poisson noise are: $A = 1$; $\rho = 1$. The length scale of the domain $[-L, L]$ is $2L = 20$. The number of allowed sites is then 20.

particle arrives on x at time t having done n cycles on the torus. When the walk is ergodic the time of recurrence goes to infinity and only the term $n = 0$ is present in the sum for each allowed value of x.

In figure 2 we have plotted the probability (2.22) as a function of time for two different allowed sites x. We have chosen the parameters ($L = 10$, $A = 1$) in such a way as to get 20 allowed sites x_i; that is why $\mathcal{P}_x(t \to \infty)$ goes to the value 0.05. Note that the probability $\mathcal{P}_{x_0}(t \to 0)$ (corresponding to the initial condition x_0) goes to one, as is expected. From figure 2 we can see that these functions have an oscillatory decreasing behaviour in time. The period of oscillation is approximately the rate between the recurrence $\mathcal{T}(A, L)$ and the hopping per unit of time ρ. Furthermore, we see that both probabilities oscillate quite similarly but with a phase shift.

The intensity of the arriving pulses A and the length of the ring L are very important parameters on the dynamics of SP $\boldsymbol{X}(t)$. If the length of the ring is decreased, or if the intensity is increased, the oscillatory behaviour on $\mathcal{P}_x(t)$ can disappear. This is because the number of allowed points is decreased, and so the equilibrium distribution $P(x, t \to \infty)$ is reached faster without any oscillatory behaviour. This is an interesting result which is harder to derive from (2.22). In fact, $\mathcal{P}_x(t)$ has a very highly nonlinear dependence on the parameters L and A. It is interesting to compare these results with the next example: a transport process in the presence of a radioactive noise, where the stationary distribution $P(x, t \to \infty)$ is not unique.

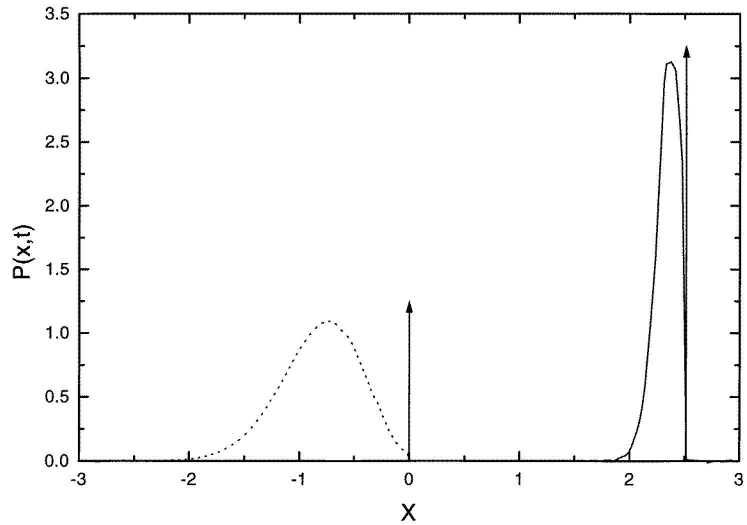

Figure 3. Probability distribution of the SP $\boldsymbol{X}(t)$ for two values of time $t = 0.125$ (solid curve) and $t = 0.3$ (dotted curve). The parameters of the radioactive noise are: $\beta = 1$; $\xi_0 = 20$. The initial contidion of SP $\boldsymbol{X}(t)$ is $x_0 = 0$, the domain $[-L, L]$ has periodic boundary conditions, and its length is $2L = 6$. The arrows represents the location of the rare events (Dirac deltas).

(4) Radioactive noise. From (2.17) and (B11) the 1-time probability distributions of SP $\boldsymbol{X}(t)$ reads:

$$P(x,t) = \frac{1}{2L} W(x) \sum_{m=-\infty}^{+\infty} \exp -\mathrm{i}\frac{m\pi}{L}(x - x_0) \times \left[\beta \frac{\{\exp[(\mathrm{i}\frac{m\pi}{L} - \beta)t] - 1\}}{(\mathrm{i}\frac{m\pi}{L} - \beta)} + \exp\left[\left(\mathrm{i}\frac{m\pi}{L} - \beta\right)t\right]\right]^{\xi_0}. \tag{2.23}$$

In figure 3 we have plotted this probability distribution as a function of x, for two different times t ($= 0.125$; 0.3). In this figure we see, at short times, a packet travelling (at the same speed) behind the Dirac delta contribution (rare event). As with dichotomous noise, here this travelling Dirac delta comes from the deterministic realization of $\xi(t)$, i.e. solutions of $\dot{X}(t) = \xi_0$.

At the longest time ($t = 0.3$) this packet reappeared on the opposite side (due to the periodic BC) and then reached the initial condition x_0, but still behind the Dirac delta, which now has a very small intensity. The dynamics of this packet is similar to the movement of a front. The main difference is that here there is a dispersion, therefore its shape varies with time until the source of noise has vanished. In the asymptotic long-time limit the probability weight of the Dirac delta is null, nevertheless the asymptotic distribution $P(x, t \to \infty)$, in general, is not constant over the whole interval $\mathcal{D} \equiv [-L, L]$. This is so due to the nature of the random 'death' of the radioactive noise $\xi(t)$. Due to this fact, all the realizations of SP $\boldsymbol{X}(t)$ end with the particle, at rest, at any random position x_f (with a finite dispersion on x); but the mean value of these position $\{x_f\}$, in general, is not zero. This mean value strongly depends of the parameters of the ring and the noise (L, β, ξ_0), showing that the stationary distribution is not unique.

2.3. Reflecting boundary condition on $\mathcal{D}$

Definition 2. *Let the SDE be*

$$\frac{\mathrm{d}X_0(t)}{\mathrm{d}t} = U'(X_0) + \xi(t) \qquad X_0(t) \in [-\infty, \infty] \tag{2.24}$$

with $U'(X_0) \equiv \frac{\mathrm{d}}{\mathrm{d}X_0} U(X_0)$, $U(X_0) = U(X_0 + 2L)$ an even periodic potential, and $\xi(t)$ an arbitrary noise. The conditional probability distribution, associated to (2.24) and satisfying reflecting BC on $\mathcal{D}$ can be built up in terms of method of images [10,11] as

$$P(x,t) = W(x) \sum_{m=-\infty}^{+\infty} \{P_0(x + 4mL, t) + P_0(-x + 4mL + 2L, t)\}. \tag{2.25}$$

As before, $W(x)$ is the window function (2.3), and $P_0(x,t)$ is the corresponding probability distribution of the SDE (2.24) with natural BC.

Theorem 2. *Let $\boldsymbol{X}_0(t)$ be any stochastic realization solution of the unbounded SDE (2.24). The realizations of the window SP*

$$\boldsymbol{X}(t) = \sum_{m=-\infty}^{+\infty} W(\boldsymbol{X}_m^+(t))\boldsymbol{X}_m^+(t) + W(\boldsymbol{X}_m^-(t))\boldsymbol{X}_m^-(t) \tag{2.26}$$

where

$$\boldsymbol{X}_m^+(t) = \boldsymbol{X}_0(t) + 4mL \qquad \boldsymbol{X}_m^-(t) = -\boldsymbol{X}_0(t) + 4mL + 2L$$

satisfy reflecting BC. Furthermore, the conditional probability distribution of $\boldsymbol{X}(t)$ is given by the distribution (2.25).

Proof. The first part of the theorem is proved with the method of dynamical images. The explicit constructions of the realizations follows by using the positive and negative images $\boldsymbol{X}_m^+(t)$ and $\boldsymbol{X}_m^-(t)$. In this case it is necessary that the periodic potential $U(X_0) = U(X_0 + 2L)$ be an even function around the origin, otherwise the method of images fails. The proof of the second part is entirely analogous to that of theorem 1. □

Performing the same steps as we used to obtain the previous propositions and corollaries (for periodic BC), similar expressions for the characteristic function, conditional probability and moments can be obtained (see the general expressions in appendix A).

2.4. Applications to the generalized Wiener process: reflecting BC

In this section we study the generalized Wiener process with reflecting BC on $\mathcal{D}$. As in the previous section, we put $U = 0$ and let $\xi(t)$ be an arbitrary noise in (2.24). Because this system fulfils all the conditions of theorem 2 it applies straightforwardly.

2.4.1. The first moment. Here we are going to study the mean value of the SP (2.26) for the same class of noises $\xi(t)$ used before. From appendix A, the final expression for the first moment is:

$$\langle \boldsymbol{X}(t) \rangle = L \sum_{m=-\infty; m\neq 0}^{+\infty} \frac{(-1)^{\frac{m-1}{2}}}{(\frac{m\pi}{2})^2} \mathrm{Im}\left[G_{X_0}\left(\frac{m\pi}{2L}, t\right)\right] \qquad m \text{ odd}. \tag{2.27}$$

Once again we remark that to use this expression we only need to know the 1-time characteristic function of the SP $\boldsymbol{X}_0(t)$ with natural BC. This object has been calculated for several noises in appendix B.

Examples. *From (B4) and (2.27) we obtain with a* Gaussian colour noise

$$\langle \boldsymbol{X}(t)\rangle = 2L \sum_{m=1, m\ odd}^{+\infty} \frac{(-1)^{\frac{m-1}{2}}}{(\frac{m\pi}{2})^2} \sin\left(\frac{m\pi}{2L}x_0\right) \exp\left[\frac{-\Gamma_2}{2}\left(\frac{m\pi}{2L}\right)^2 (t + \tau_c(\mathrm{e}^{-t/\tau_c} - 1))\right]. \tag{2.28}$$

Using (B6) and (2.27) we obtain with a dichotomous noise†

$$\langle \boldsymbol{X}(t)\rangle = 2L \exp(-\lambda t) \sum_{m=1, m\ odd}^{+\infty} \frac{(-1)^{\frac{m-1}{2}}}{(\frac{m\pi}{2})^2} \sin\left(\frac{m\pi}{2L}x_0\right) \times \left[\cosh\left(\sqrt{\lambda^2 - \left(\frac{m\pi}{2L}a\right)^2} t\right) + \frac{\lambda}{\sqrt{\lambda^2 - (\frac{m\pi}{2L}a)^2}} \sinh\left(\sqrt{\lambda^2 - \left(\frac{m\pi}{2L}a\right)^2} t\right)\right]. \tag{2.29}$$

Using (B9) and (2.27) we obtain with a Poisson noise

$$\langle \boldsymbol{X}(t)\rangle = 2L \sum_{m=1, m\ odd}^{+\infty} \frac{(-1)^{\frac{m-1}{2}}}{(\frac{m\pi}{2})^2} \exp\left[\rho t\left(\cos\left(A\frac{m\pi}{2L}\right) - 1\right)\right] \sin\left[\rho t \sin\left(A\frac{m\pi}{2L}\right) + \frac{m\pi}{2L}x_0\right]. \tag{2.30}$$

Using (B11) and (2.27) we obtain with a radioactive noise

$$\langle \boldsymbol{X}(t)\rangle = L \sum_{m=-\infty; m\neq 0}^{+\infty} \frac{(-1)^{\frac{m-1}{2}}}{(\frac{m\pi}{2})^2} \mathrm{Im}\left[\left[\beta \frac{\exp t(\mathrm{i}\frac{m\pi}{2L} - \beta) - 1}{\mathrm{i}\frac{m\pi}{2L} - \beta} + \exp t\left(\mathrm{i}\frac{m\pi}{2L} - \beta\right)\right]^{\xi_0} \exp \mathrm{i}\frac{m\pi}{2L}x_0\right] \qquad m \text{ odd}. \tag{2.31}$$

In all the examples $x_0 \in \mathcal{D}$ is the initial condition of the SP $\boldsymbol{X}(t)$ with reflecting BC.

Figure 4 shows the comparison of the first moment $\langle \boldsymbol{X}(t)\rangle$ for two different structures of noise, and different noise correlation times ($\tau_c = 0.1$; 10 for the Gaussian colour-noise, and $\lambda^{-1} = 1$; 10 for the dichotomous case—solid curves).

In presence of a *Gaussian colour noise* $\xi(t)$, a typical monotonous relaxation for $\langle \boldsymbol{X}(t)\rangle$ is obtained, i.e. (2.28). For larger τ_c the relaxation gets slower, and this is so because the particle has more memory to remember its initial condition, and therefore a plateau appears at very short times. This phenomenon is a non-Markovian effect of the SP $\boldsymbol{X}(t)$. In the limit of a very long correlation noise (i.e. $\tau_c \gg 1$) the shape of the function $\langle \boldsymbol{X}(t)\rangle$ is almost constant during a large period of time $\mathcal{O}(\tau_c)$. In contrast, if we increase the noise parameter Γ_2, the relaxation gets faster. This is also an intuitive result, because when the intensity of the noise is increased the whole domain $\mathcal{D}$ is visited much faster (the SP $\boldsymbol{X}(t)$ spreads faster). In any case, by using a *Gaussian* noise the relaxation is a monotonous decreasing function of time. We remark that this is not the case if the structure of the noise is non-Gaussian as in (2.29).

Basically, if the correlation time of the *dichotomous noise* λ^{-1} is small, the relaxation of $\langle \boldsymbol{X}(t)\rangle$ is the same as with a Gaussian noise (we have used different initial conditions: $x_0 = \pm 2$ in order to highlights both curves). Nevertheless, when the noise correlation time λ^{-1} is sufficiently long, owing to the nature of the *dichotomous noise*, there will be rare events due to the *special* realizations $\xi(t) = \pm a$ (with a probability weight $\propto \mathrm{e}^{-\lambda t}$) that survive for

† This expression can be obtained from formulae (19) of [7] with the following steps: a translation of length L, changes $L \to 2L$, $1/2T \to \lambda$, $c \to a$, and take $\alpha = \frac{1}{2}$. Furthermore, formulae (19) of [7] can be obtained from (A7) (with dichotomous noise) with the same procedure.

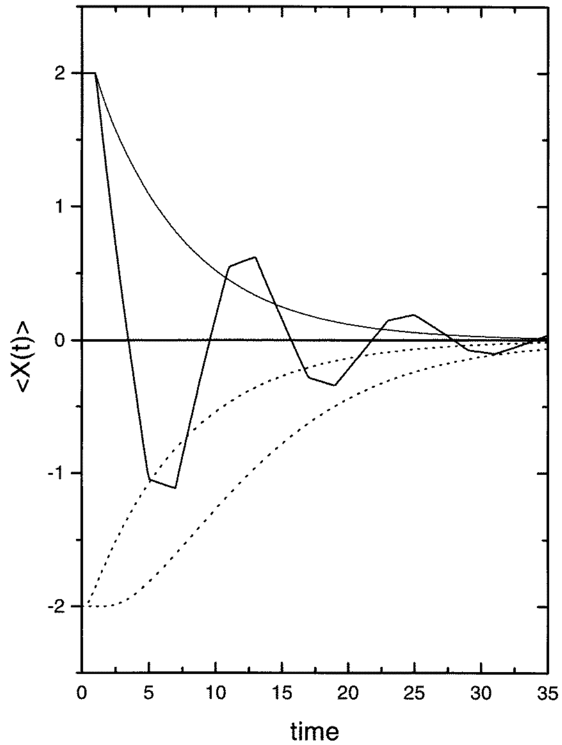

Figure 4. First moment of the SP $\boldsymbol{X}(t)$ as a function of time t for two different noises $\xi(t)$. Dotted curves correspond to the presence of a Gaussian non-white noise with parameters: $\Gamma_2 = 1$; $\tau_c = 0.1$ and $\tau_c = 10$ (long plateau); there the initial condition of SP $\boldsymbol{X}(t)$ is $x_0 = -2$. Solid curves correspond to the presence of a dichotomic noise with parameters: $a = 1$; $\lambda^{-1} = 1$ and $\lambda^{-1} = 10$ (with oscillations); there the initial condition is $x_0 = 2$. In all these plots the boundary conditions are reflecting on $[-L, L]$ and the length of the domain is $2L = 6$.

a long time. These *deterministic* noise realizations are the responsibility of the oscillations in the mean value of SP $\boldsymbol{X}(t)$. If we increase the parameter λ, the correlation time gets shorter and shorter and this can 'kill' any reminiscence of the *deterministic realizations* $\xi(t) = \pm a$. Therefore, there is a threshold $\lambda^{-1} \leqslant$ *critical value* below which the oscillatory behaviour of $\langle \boldsymbol{X}(t) \rangle$ does not exist any more, and then the relaxation of $\langle \boldsymbol{X}(t) \rangle$ is the same as in the presence of a Gaussian noise $\xi(t)$. Finally we note that there always exists—at short times—a plateau in the behaviour of $\langle \boldsymbol{X}(t) \rangle$. This phenomenon comes from the symmetry of the noise and the fact that no realizations spread more than the deterministic realizations. Therefore, until any of the Dirac deltas has reached some frontier, the behaviour of $\langle \boldsymbol{X}(t) \rangle$ is the same as with natural BC, i.e. it remains constant.

In figure 5 we show $\langle \boldsymbol{X}(t) \rangle$ in the presence of a *Poisson noise*, i.e. (2.30). In this figure we have used the initial condition $x_0 = 0$ and we have plotted three different cases by changing the amplitude A of the noise pulse. For this case we see that at short times the first-moment $\langle \boldsymbol{X}(t) \rangle$ is an increasing function of time; this is so because the noise $\xi(t)$ is a sequence of random pulses (with intensity A) arriving in the same direction. After this transient has finished, the relaxation shows a decaying oscillatory behaviour, but now this oscillatory behaviour is always present and is due to BC on $\mathcal{D}$. For a fixed length L but increasing intensity A, the amount of allowed points (the set $\{x_i\}$) is reduced, therefore the stationary distribution $P(x, t \to \infty)$ is

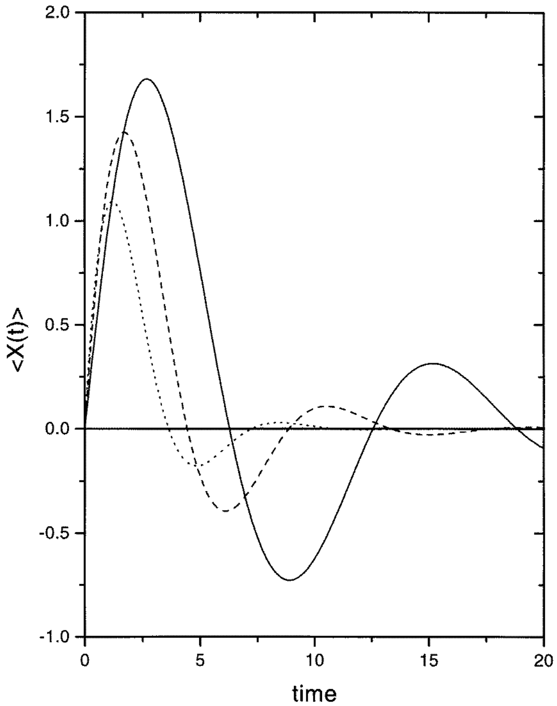

Figure 5. First moment of the SP $\boldsymbol{X}(t)$ as a function of time for a Poisson noise $\xi(t)$. The parameters are: $\rho = 1$; $A = 1$ (solid curve); $A = 1.5$ (dashed curve) and $A = 2$ (dotted curve). The initial condition is $x_0 = 0$ and the boundary conditions are reflecting on $[-L, L]$. The length of the domain is $2L = 6$.

reached faster and this leads to the fact that the relaxation of $\langle \boldsymbol{X}(t) \rangle$ goes to zero faster too. This fact can also be seen from our plots.

In figure 6 we show the case when the noise is a *radioactive* one, i.e. (2.31). In this figure we have used the initial condition $x_0 = 0$ and we have plotted three different cases by changing the initial noise quantity ξ_0 (= 5; 10; 20), i.e. the initial amount of atoms in the jargon of the radioactive noise. The increasing behaviour—at short times—of the function $\langle \boldsymbol{X}(t) \rangle$ is due to the fact that the radioactive noise is a constant force that at random times decreases a finite quantity (non-symmetric noise). If the number ξ_0 is small, the source of noise could vanish before the particle (the SP $\boldsymbol{X}(t)$) can reach the frontier of $\mathcal{D}$, therefore the behaviour of $\langle \boldsymbol{X}(t) \rangle$ would be similar to the case in the presence of natural BC. If the quantity ξ_0 is large enough, the effect of the reflecting BC starts to be important and the behaviour of $\langle \boldsymbol{X}(t) \rangle$ turns out to be oscillatory. This change in the behaviour—from non-oscillatory to oscillatory—can also be obtained by changing the correlation time of the radioactive noise β^{-1}. The analysis turns out to be equivalent to the one coming from dichotomous noise: by increasing β^{-1} the *deterministic* realization $\xi(t) = \xi_0$ starts to be important (now there will be only one Dirac delta), therefore in the limit $\beta^{-1} \to \infty$ the behaviour of $\langle \boldsymbol{X}(t) \rangle$ is a *saw tooth*. When the correlation time β^{-1} gets shorter than a *critical value* the relaxation does not show any oscillatory behaviour owing to the fast loss of the deterministic realization.

The long-time limit of the first moment of $\boldsymbol{X}(t)$ is non-null because the probability distribution, in general, is not uniform on $\mathcal{D}$. In particular, with increasing ξ_0 the asymptotic long-time limit of $\langle \boldsymbol{X}(t) \rangle$ gets closer to zero. This is so because by increasing this quantity, the

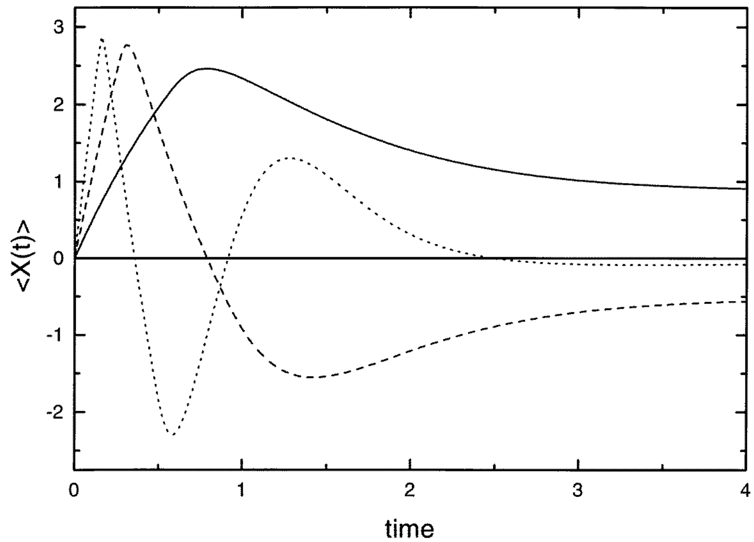

Figure 6. First moment of the SP $\boldsymbol{X}(t)$ as a function of time for a radioactive source of noise $\xi(t)$, with parameter $\beta = 1$. In all these cases the initial condition of SP $\boldsymbol{X}(t)$ was $x_0 = 0$. The solid curve corresponds to a presence of $\xi_0 = 5$ initial atoms. The dashed curve corresponds to $\xi_0 = 10$, and the dotted curve to the case $\xi_0 = 20$. In all these plots the boundary conditions are reflecting on $[-L, L]$ and the length of the domain is $2L = 6$.

'live-time' of the noise is larger, and due to the BC this fact increases the dispersion of the final position x_f of the particle. Then the mean value of these random positions $\{x_f\}$ approaches zero. A similar phenomenon was also explained with periodic BC, see (2.23).

3. Semi-infinite domain

In this section we study a generalized Langevin equation with a reflecting boundary condition at the coordinate origin, so the domain of interest will be $[0, \infty]$. The characterization of this case can be made from theorem 2, with the following steps: a translation of length L, change L to $2L$ and take the limit $L \to \infty$. Alternatively, all results can be obtained from the explicit construction of the realizations satisfying reflecting BC in the origin. The constructions of these realizations $\boldsymbol{Y}(t)$ follows by the use of a negative mirror image—around the origin—of the positive one

$$\boldsymbol{Y}(t) = \Theta(\boldsymbol{Y}_0(t))\boldsymbol{Y}_0(t) - \Theta(-\boldsymbol{Y}_0(t))\boldsymbol{Y}_0(t) \tag{3.1}$$

where $\Theta(y)$ is the step function and $\boldsymbol{Y}_0(t)$ are the realization of the unbounded process

$$\frac{\mathrm{d}Y_0(t)}{\mathrm{d}t} = U'(Y_0) + \xi(t) \qquad Y_0(t) \in [-\infty, \infty]. \tag{3.2}$$

From this procedure it is obvious that now it is required that the potential $U(Y_0)$ be an even function around the origin, otherwise the method of image fails.

From (3.1), use the following result for the mean value:

$$\langle \boldsymbol{Y}(t)\rangle = \frac{-1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}k}\langle \exp \mathrm{i}kY_0(t)\rangle\,\mathrm{d}k = \frac{-1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}k}G_{Y_0}(k,t)\,\mathrm{d}k. \tag{3.3}$$

3.1. Application to the generalized Ornstein–Uhlenbeck process

Here we are going to study the generalized Ornstein–Uhlenbeck process on a semi-infinite domain, with reflecting BC at the origin of coordinates. Thus we take $U' \to U'(V) = -\gamma_1(t)V$ and $\xi \to \gamma_2(t)\xi(t)$ in (3.2). So here we propose to study the non-autonomous SDE:

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) = -\gamma_1(t)V + \gamma_2(t)\xi(t). \tag{3.4}$$

As usual we adopt that the *random force* is any noise completely characterized by its functional $G_\xi([k(t)])$ and $\gamma_2(t)$ and $\gamma_1(t)$ are sure functions of the time. A complete characterization of the unbounded SP $\boldsymbol{V}(t)$ follows from proposition 2 of paper I. There we proved that

$$G_V([Z(t)]) = \mathrm{e}^{+\mathrm{i}k_0 V(0)}\, G_\xi\left(\left[\gamma_2(t)\int_t^\infty \mathrm{e}^{\int_{t'}^{t}\gamma_1(s)\,\mathrm{d}s} Z(t')\,\mathrm{d}t'\right]\right) \tag{3.5}$$

where $k_0 = \int_0^\infty Z(s)\exp(-\int_0^s \gamma_1(t)\,\mathrm{d}t)\,\mathrm{d}s$, and $\xi(t)$ is characterized by its functional.

Now we want to exemplify the application of (3.3) to evaluate $\langle \boldsymbol{V}(t)\rangle$ when $\xi(t)$ is a Gaussian colour noise (with arbitrary correlation $\langle \xi(t_1)\xi(t_2)\rangle$, for details see appendix B). After some algebra, the exact first moment reads†

$$\langle \boldsymbol{V}(t)\rangle = 2\sqrt{\frac{\sigma(t)}{\pi}}\exp\left(\frac{-B(t)^2}{4\sigma(t)}\right) + B(t)\mathrm{erfc}\left(\frac{-B(t)}{2\sqrt{\sigma(t)}}\right) \qquad t > 0 \tag{3.6}$$

$\langle \boldsymbol{V}(0)\rangle = B(0) > 0$, and where $\sigma(t)$ and $B(t)$ are functions of time given by

$$\begin{aligned}
\sigma(t) &\equiv \frac{\Gamma_2}{2}\int_0^t\int_0^t \mathrm{d}s_1\,\mathrm{d}s_2\,\gamma_2(s_1)\gamma_2(s_2)\mathrm{e}^{\int_t^{s_1}\gamma_1(s')\,\mathrm{d}s'}\mathrm{e}^{\int_t^{s_2}\gamma_1(s')\,\mathrm{d}s'}\langle\xi(s_1)\xi(s_2)\rangle \\
B(t) &\equiv V(0)\mathrm{e}^{-\int_0^t \gamma_1(s')\,\mathrm{d}s'}.
\end{aligned} \tag{3.7}$$

One of the simplest cases to analyse is when $\gamma_1(t) = \gamma$, $\gamma_2(t) = 1$ and the Gaussian noise has a short-range correlation, i.e. $\langle\xi(t)\xi(t')\rangle = \frac{1}{2\tau_c}\exp(-|t-t'|/\tau_c)$. In this particular autonomous case, the first moment $\langle \boldsymbol{V}(t)\rangle$ is given by (3.6), with $B(t) = V(0)\mathrm{e}^{-\gamma t}$ and

$$\sigma(t) = \frac{\Gamma_2}{4\gamma}\left\{\frac{1-\mathrm{e}^{-2\gamma t}}{1-\tau_c^2\gamma^2} + \frac{\gamma\tau_c(2\mathrm{e}^{-(\gamma+\frac{1}{\tau_c})t} - \mathrm{e}^{-2\gamma t} - 1)}{1-\tau_c^2\gamma^2}\right\}. \tag{3.8}$$

This expression is valid for any $\tau_c \geqslant 0$ and $\gamma \geqslant 0$. Note that both summands in (3.8) have non-Markovian contributions. In the limit $\tau_c = 0$ this expression coincides with the well known result that could be obtained from Fokker–Planck dynamics: $\sigma(t) = \frac{\Gamma_2}{4\gamma}(1-\mathrm{e}^{-2\gamma t})$. An interesting result is the fact that the non-Markovian effect ($\tau_c \neq 0$) shifts—towards smaller values—the long-time limit of the first moment $\langle \boldsymbol{V}(t\to\infty)\rangle = \sqrt{\frac{\Gamma_2}{\gamma\pi}\frac{1}{1+\gamma\tau_c}}$. On the other hand, the presence of a Gaussian colour noise induces a resonant-like behaviour at the correlation value $\tau_c = 1/\gamma$. This can be seen by taking the suitable limit in (3.8), which is—of course—a pure non-Markovian effect.

3.2. Application to the generalized Wiener process

Here we want to show another possible application of (3.3), but when the SDE is $\dot{X}(t) = \xi(t)$. First we analyse the case when $\xi(t)$ is a Gaussian colour noise. In appendix B we have given

† We have used $\mathcal{P}\int_{-\infty}^{\infty}(\mathrm{i}x)^\nu\exp(-\beta^2x^2 - \mathrm{i}qx)\,\mathrm{d}x = 2^{-\nu/2}\sqrt{\pi}\,\beta^{-\nu-1}\exp\left(\frac{-q^2}{8\beta^2}\right)D_\nu\left(\frac{q}{\beta\sqrt{2}}\right)$, for $\mathrm{Re}\,[\beta] > 0$, where $D_\nu(z)$ is the Dyson special function.

the characteristic function for the unbounded SP $\boldsymbol{X}_0(t)$. Thus from (3.3) and (B4) the first moments reads

$$\langle \boldsymbol{X}(t)\rangle = 2\sqrt{\frac{\sigma(t)}{\pi}}\exp\left(\frac{-X(0)^2}{4\sigma(t)}\right) + X(0)\mathrm{erfc}\left(\frac{-X(0)}{2\sqrt{\sigma(t)}}\right) \tag{3.9}$$

where

$$\sigma(t) = \frac{\Gamma_2}{2}(t + \tau_c(\mathrm{e}^{-t/\tau_c} - 1)) \tag{3.10}$$

and $X(0) > 0$ is the initial condition. We note that this result can also be obtained from (3.6), by putting $\gamma_1(t) = 0$, $\gamma_2(t) = 1$ (or alternatively from (3.8) taking the limit $\gamma \to 0$). Equation (3.10) shows that the non-Markovian character ($\tau_c \neq 0$) in the relaxation of $\langle \boldsymbol{X}(t)\rangle$ survives only for a time of $\mathcal{O}(\tau_c)$, as was expected. Therefore, in the long-time limit, Wiener's root square scaling is obtained, $\langle \boldsymbol{X}(t)\rangle \sim \sqrt{t}$.

The Wiener particle can easily be reobtained by taking the limit $\tau_c \to 0$ in (3.9). In this case our result, of course, coincides with the one obtained from the diffusion equation: $\partial_t P(x,t) = \frac{\Gamma_2}{2}\partial_x^2 P(x,t)$, considering a reflecting boundary condition on $x = 0$.

Finally, we analyse the particular case when $\xi(t)$ has a non-Gaussian structure, for example when it is the sum of two statistical independent Poisson noises $\pm\xi(t)$ with intensity A and with average numbers of pulses ρ. In the limit $\rho \to \infty$ and $A \to 0$ such that $\rho A^2 \to \Gamma_2/2 =$ finite this noise reduces to a zero-mean Gaussian white noise. From appendix B it is simple to see that the corresponding characteristic function of the unbounded generalized Wiener process is

$$G_{X_0}(k,t) = \mathrm{e}^{+\mathrm{i}kX_0(0)}\exp[2\rho t(\cos Ak - 1)]. \tag{3.11}$$

Now, from the application (3.3) (for the case $U = 0$) and using (3.11), the exact first moment $\langle \boldsymbol{X}(t)\rangle$ is given by

$$\langle \boldsymbol{X}(t)\rangle = 2A\rho t\exp(-2\rho t)[I_0(2\rho t) + I_1(2\rho t)] \tag{3.12}$$

where $I_0(z)$, $I_1(z)$ are modified Bessel functions and we have used the initial condition $\boldsymbol{X}(0) = 0$. From this expression it is possible to see that—induced by the non-Gaussian structure of the noise $\xi(t)$—the behaviour of $\langle \boldsymbol{X}(t)\rangle$ shows a very long transient, but its asymptotic long-time limit gives $\langle \boldsymbol{X}(t)\rangle \sim A\sqrt{\rho t}$. We remark that in this analysis we have not taken the Gaussian limit, which is why the parameters ρ and A are still present in the relaxation of the first moment. However, if the intensity of the arriving pulses is very large and its density is small (i.e. in the limit $A \to \infty$ and $\rho \to 0$ with $\rho A \to$ constant), from (3.12) a (temporal) linear behaviour $\langle \boldsymbol{X}(t)\rangle \sim t$ is obtained, in contrast with the familiar Wiener root-square scaling. In summary, a non-Gaussian but white-noise $\xi(t)$ like the Poisson one, only induces a long transient, but Wiener's universal scaling $\langle \boldsymbol{X}(t)\rangle \sim \sqrt{t}$ is still present at long time. We expect that other structures of noise could drive to similar conclusions. Only long-range noise correlation will change the long-time scaling of $\langle \boldsymbol{X}(t)\rangle$.

4. Conclusions

The goal addressed in this paper—by proving our theorems—has been to be able to give an exact closed characterization of a large class of finite non-Markovian processes that follows the evolution: $\frac{\mathrm{d}X(t)}{\mathrm{d}t} = U'(X) + \xi(t)$, when $\xi(t)$ is an arbitrary noise.

Assuming that the functional $G_{X_0}([Z(t)])$ of the Langevin-like unbounded SP $\boldsymbol{X}_0(t)$ is known, the generalized process $\boldsymbol{X}(t)$ on the finite domain $\mathcal{D} \equiv [-L, L]$ has been completely characterized for periodic and reflecting BC on $\mathcal{D}$. We have given exact expressions for the whole Kolmogorov hierarchy, from which all the information concerning mean values, stationary distributions, correlations functions, etc. can be analysed.

Owing to the fact that the SP $\boldsymbol{X}_0(t)$ could be non-Markovian, there is not a clear partial differential equation available for its n-time probability distribution. Thus, in order to characterize the SP $\boldsymbol{X}(t)$ we have explicitly constructed stochastic realizations that satisfy the corresponding BC. This was made by using a method of dynamical images by defining the *window SP* presented through theorems 1 and 2. We remark that this method only works when the corresponding periodicity conditions, on the potential U, are fulfilled.

We have exemplified the method putting $U = 0$. This corresponds to the generalized Wiener process where the unbounded characteristic functional is available from paper I. There we have introduced a fairly general method based upon knowing the characteristic functional of the arbitrary noise $G_\xi([k(t)])$ (see also appendix B of this paper).

In order to find the interplay between non-Gaussian noises and the effects of the frontier of $\mathcal{D}$, some examples with non-Gaussian structures of noise have been worked out. In particular, we have used dichotomous noise, Poisson noise and radioactive noise. It is interesting to note that as with the dichotomous noise [7], the radioactive noise could have important medical applications.

With periodic BC we have studied the 1-time probability distribution. This was shown in figures 1 and 3, from which there are some results to remark on. One is the presence of *rare events* corresponding to the permanence of δ contributions in the probability distribution (for $t > 0$). The occurrence of these rare events also appears in the unbounded case owing the existence of deterministic realizations. The relation between the motion of these *boundaries* in the probability distribution (δ contributions) and the structure of the noise appearing in its corresponding SDE has been pointed out. On the other hand, the oscillatory behaviour (in the presence of Poisson noise) and the presence of travelling packets (in the presence of radioactive noise) in the probability distribution $P(x, t)$ are shown to be consequences of the highly non-Gaussian character of the noise.

With reflecting BC, the emphasis was on the characterization of the mean values, i.e. by analysing some exact results for the moments of $\boldsymbol{X}(t)$. Many aspects of the evolution are evident in figures 4–6. One of the most important is the oscillatory behaviour of the moments, a phenomenon that never can be reached with a Gaussian (colour or white) noise.

The generalized Ornstein–Uhlenbeck process on a semi-infinite domain and with reflecting BC at the coordinate origin has also been analysed.

We emphasize that the present formulation is exact and provides a systematic starting point to obtain higher-order moments, and the whole Kolmogorov hierarchy for a large class of non-Markovian process on *finite domains*. We remark that in this analysis the only object which it is necessary to know is the characteristic functional of the unbounded process $G_{X_0}([Z(t)])$. In this paper we have used previous results (from paper I) on functional techniques to solve a related problem but now in the presence of non-natural BC.

Acknowledgments

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Appendix A. The Kolmogorov hierarchy

We remarked in paper I that a complete characterization of a non-Markovian SP $\boldsymbol{X}(t)$ demands the knowledge of the whole Kolmogorov hierarchy, i.e. the n-time joint probability distribution

$P(x_1, t_1; x_2, t_2; \ldots; x_n, t_n)$, or equivalently the characteristic functional $G_X([Z(t)])$. From this functional all the n-time moments can be calculated by nth order functional differentiation (and, of course, their n-times cumulants too). Here we want to emphasize that if we know a closed expression of the functional $G_X([Z(t)])$, this fact implies knowledge of the whole Kolmogorov hierarchy. This issue is simple to realize by introducing the Fourier representation of the δ-function in the following expression [1]:

$$P(x_1, t_1; x_2, t_2; \ldots; x_n, t_n) = \langle \delta(\boldsymbol{X}(t_1) - x_1)\delta(\boldsymbol{X}(t_2) - x_2) \ldots \delta(\boldsymbol{X}(t_n) - x_n)\rangle.$$

For example, the 1-time probability distribution $P(x_1, t_1)$ is given by quadrature in terms of the $G_X([Z(t)])$ evaluated with the test function $Z(t) = k_1\delta(t - t_1)$. In general, we can invert the characteristic functional by introducing the n-dimensional Fourier transform

$$P(x_1, t_1; x_2, t_2; \ldots; x_n, t_n) = \frac{1}{(2\pi)^n} \int \ldots \int \mathrm{d}k_1 \ldots \mathrm{d}k_n \, \exp\left(- \mathrm{i} \sum_{i=1}^{n} k_i x_i \right) \times [G_X([Z(t)])]_{Z(t)=k_1\delta(t-t_1)+\cdots+k_n\delta(t-t_n)}. \tag{A1}$$

This formula is sometimes remarkably simple to work out in comparison with the calculation of the n-time joint probability distribution given in terms of a series expansion that we made in paper I.

Appendix A.1. Non-Markovian processes with special boundary conditions

In this case we are not able to construct the characteristic functional of the processes, nevertheless, we can construct for each natural n the n-time joint probability distribution.

First of all, note that what we need in (A1) is the n-time characteristic function of the process, i.e.

$$G_X(k_1, t_1; k_2, t_2; \ldots; k_n, t_n) = [G_X([Z(t)])]_{Z(t)=k_1\delta(t-t_1)+\cdots+k_n\delta(t-t_n)} \tag{A2}$$

i.e. we have information on the n-time moment

$$\langle \boldsymbol{X}(t_1)\boldsymbol{X}(t_2) \ldots \boldsymbol{X}(t_n)\rangle = (-\mathrm{i})^n \, \frac{\partial^n}{\partial k_1 \partial k_2 \cdots \partial k_n} G_X(k_1, t_1; k_2, t_2; \ldots; k_n, t_n)\bigg|_{k_1=k_2=\ldots k_n=0}. \tag{A3}$$

Now it is simple, using our previous propositions, to write down this object for a finite system characterized with some special BC on the domain $\mathcal{D} = [-L, L]$. For example, if the BC is periodic on $\mathcal{D}$, proposition 3 tells how to built up the 1-time characteristic function if we know the characteristic function of the unbounded SP $\boldsymbol{X}_0(t)$, i.e. $G_{X_0}(k_1, t_1) = \langle \exp \mathrm{i}k_1\boldsymbol{X}_0(t_1)\rangle$. The same procedure is easy to generalize for any n-time characteristic function, using the property (A3) and following the same steps as in propositions 2 and 3. The n-time characteristic function of a finite system with periodic BC results:

$$G_X(k_1, t_1; k_2, t_2; \ldots; k_n, t_n) = \sum_{m_1=-\infty}^{+\infty} \ldots \sum_{m_n=-\infty}^{+\infty} \frac{\sin(m_1\pi - k_1L)}{(m_1\pi - k_1L)} \ldots \frac{\sin(m_n\pi - k_nL)}{(m_n\pi - k_nL)} \times G_{X_0}\left(\frac{m_1\pi}{L}, t_1; \ldots; \frac{m_n\pi}{L}, t_n\right) \tag{A4}$$

where

$$G_{X_0}\left(\frac{m_1\pi}{L}, t_1; \ldots; \frac{m_n\pi}{L}, t_n\right) = [G_{X_0}([Z(t)])]_{Z(t)=\frac{m_1\pi}{L}\delta(t-t_1)+\cdots+\frac{m_n\pi}{L}\delta(t-t_n)}$$

follows from the unbounded characteristic functional.

Now introducing (A4) in (A1) the n-time joint probability results:

$$P(x_1,t_1;x_2,t_2;\ldots;x_n,t_n)=\frac{1}{(2L)^n}\left[\prod_{i=1}^{n}W(x_i)\right]\sum_{m_1=-\infty}^{+\infty}\cdots\sum_{m_n=-\infty}^{+\infty}\exp\left(-\mathrm{i}\sum_{i=1}^{n}\frac{m_i\pi}{L}x_i\right)\times G_{X_0}\left(\frac{m_1\pi}{L},t_1;\ldots;\frac{m_n\pi}{L},t_n\right). \tag{A5}$$

If the BC are reflecting, then by making similar steps as the ones made to arrive to equation (A4), the following expression for the n-time characteristic function can be obtained:

$$G_X(k_1,t_1;k_2,t_2;\ldots;k_n,t_n)=\frac{1}{2^n}\sum_{m_1=-\infty}^{+\infty}\cdots\sum_{m_n=-\infty}^{+\infty}\frac{\sin(\frac{m_1\pi}{2}-k_1L)}{(\frac{m_1\pi}{2}-k_1L)}\cdots\frac{\sin(\frac{m_n\pi}{2}-k_nL)}{(\frac{m_n\pi}{2}-k_nL)}\times\sum_{\sigma_1=\pm1}\cdots\sum_{\sigma_n=\pm1}(-1)^{(\sum_{i=1}^{n}m_i\delta_{(-1,\sigma_i)})}G_{X_0}\left(\sigma_1\frac{m_1\pi}{2L},t_1;\ldots;\sigma_n\frac{m_n\pi}{2L},t_n\right). \tag{A6}$$

Therefore, the n-time probability distribution is in this case

$$P(x_1,t_1;x_2,t_2;\ldots;x_n,t_n)=\frac{1}{(4L)^n}\left[\prod_{i=1}^{n}W(x_i)\right]\sum_{m_1=-\infty}^{+\infty}\cdots\sum_{m_n=-\infty}^{+\infty}\exp\left(-\mathrm{i}\sum_{i=1}^{n}\frac{m_i\pi}{2L}x_i\right)\times\sum_{\sigma_1=\pm1}\cdots\sum_{\sigma_n=\pm1}(-1)^{(\sum_{i=1}^{n}m_i\delta_{(-1,\sigma_i)})}G_{X_0}\left(\sigma_1\frac{m_1\pi}{2L},t_1;\ldots;\sigma_n\frac{m_n\pi}{2L},t_n\right). \tag{A7}$$

We remark that (A4)–(A7) are exact expressions, valid whatever the structure of the noise $\xi(t)$ is. From these expressions the whole Kolmogorov hierarchy and any n-time moment is available. This fact follows from nth partial differentiation of the n-time characteristic functions.

Appendix B. The generalized Wiener process on unbounded domains

In this appendix we show some results concerning the generalized Wiener process with natural BC and for different noises. A more formal presentation is available in paper I.

Let $G_{X_0}([k(t)]) = \langle\exp\mathrm{i}\int_0^\infty k(t)X_0(t)\,\mathrm{d}t\rangle$ be the characteristic functional of the generalized Wiener SP $\boldsymbol{X}_0(t)$ in an unbounded domain, i.e. characterized by the SDE: $\frac{\mathrm{d}}{\mathrm{d}t}X_0(t)=\xi(t), X_0(t)\in[-\infty,\infty]$; with $t\in[0,\infty]$, where $\xi(t)$ is some arbitrary noise characterized by its functional $G_\xi([k(t)])$, and $k(t)$ is restricted to those real test functions that vanish for sufficiently large t. Proposition 3 of paper I tells that the generalized SP $\boldsymbol{X}_0(t)$ (with natural BC) and with a sure (non-random) initial condition $X_0(0)$ is completely characterized by the functional

$$G_{X_0}([Z(t)])=\mathrm{e}^{+\mathrm{i}k_0X_0(0)}G_\xi\left(\left[\int_t^\infty Z(s)\,\mathrm{d}s\right]\right) \tag{B1}$$

where k_0 is given by $k_0=\int_0^\infty Z(s)\,\mathrm{d}s$. The generalized *time-dependent* Wiener process, i.e. the one characterized by the non-autonomous SDE $\dot{X}_0(t)=\gamma(t)\xi(t)$ where $\gamma(t)$ is a sure function of time, is characterized by the functional

$$G_{X_0}([Z(t)])=\mathrm{e}^{+\mathrm{i}k_0X_0(0)}G_\xi\left(\left[\gamma(t)\int_t^\infty Z(t')\,\mathrm{d}t'\right]\right) \tag{B2}$$

where k_0 is as before: $k_0=\int_0^\infty Z(s)\,\mathrm{d}s$. This follows from propositions 2 and 3 of paper I.

The 1-time characteristic function of the unbounded SP $\boldsymbol{X}_0(t)$, denoted by $G_{X_0}(k_1,t_1)=\langle\exp\mathrm{i}k_1X_0(t_1)\rangle$, follows from evaluating (B1) (or from (B2) for the *time-dependent* case) with the test function $Z(t)=k_1\delta(t-t_1)$. From this characteristic function, the 1-time probability distribution $P(x_1, t_1)$ can be obtained by Fourier inversion. In what follows we exemplify these objects for different structures of noise $\xi(t)$.

Example 1. *Let* $\xi(t)$ *be a zero-mean* Gaussian colour noise *with an arbitrary correlation* $\langle\xi(t)\xi(t')\rangle$. *Its characteristic functional is* $G_\xi([k(t)]) = \exp\frac{-\Gamma_2}{2}\int_0^\infty\int_0^\infty k(s_1)k(s_2)\langle\xi(s_1)\xi(s_2)\rangle\,\mathrm{d}s_1\,\mathrm{d}s_2$. *Taking* $\langle\xi(t)\xi(t')\rangle = \frac{1}{2\tau_c}\exp(-|t-t'|/\tau_c)$ *and using (B1), the characteristic functional of the unbounded SP* $\boldsymbol{X}_0(t)$ *with* $t\in[0,\infty]$ *can be written in the form:*

$$G_{X_0}([Z(t)]) = \mathrm{e}^{+\mathrm{i}k_0X_0(0)}\exp\left[\frac{-\Gamma_2}{2}\int_0^\infty \mathrm{d}s_1\int_0^\infty \mathrm{d}s_2\int_{s_1}^\infty Z(s')\,\mathrm{d}s'\int_{s_2}^\infty \times Z(s'')\,\mathrm{d}s''\frac{1}{2\tau_c}\exp(-|s_1-s_2|/\tau_c)\right]. \tag{B3}$$

Introducing the test function $Z(t) = k_1\delta(t-t_1)$ *in (B3) we get the 1-time characteristic function of the unbounded SP* $\boldsymbol{X}_0(t)$

$$G_{X_0}(k_1,t_1) = \exp\left[\frac{-\Gamma_2}{2}k_1^2[t_1+\tau_c(\mathrm{e}^{-t_1/\tau_c}-1)] + \mathrm{i}k_1X_0(0)\right]. \tag{B4}$$

From this formula, the 1-time probability distribution $P(x_1,t_1)$ *is given by*

$$P(x_1,t_1) = [4\pi\sigma(t_1)]^{-1/2}\exp[-(x_1-X_0(0))^2/4\sigma(t_1)] \tag{B5}$$

where $\sigma(t_1)\equiv\frac{\Gamma_2}{2}(t_1+\tau_c(\mathrm{e}^{-t_1/\tau_c}-1))$. *Note from (B4) or (B5) that for a finite time t the scale-invariance* $X(t) = \frac{1}{\sqrt{\Lambda}}X(\Lambda t)$, *familiar from the Wiener process, is broken for any* $\tau_c\neq 0$. *The Gaussian white noise* $\xi(t)$ *case can trivially be reobtained by taking the limit* $\tau_c\to 0$. *In this limit (B5) gives a result which could be also obtained from the well known diffusion equation* $\partial_t P(x,t) = \frac{\Gamma_2}{2}\partial_x^2 P(x,t)$.

Example 2. *A* dichotomous noise $\xi(t)$ *has not a closed expression for its characteristic functional* $G_\xi([k(t)])$*; this functional can only be written in terms of an infinite series [6]. Nevertheless, closed expressions for the 1-time characteristic function and the 1-time probability distribution can be found [1,13]:*

$$G_{X_0}(k_1,t_1) = \exp(-\lambda t_1 + \mathrm{i}k_1X_0(0))\left[\cosh(\gamma_{k_1}t_1) + \frac{\lambda}{\gamma_{k_1}}\sinh(\gamma_{k_1}t_1)\right] \tag{B6}$$

where $\gamma_{k_1} = \sqrt{\lambda^2-(k_1a)^2}$ *and*

$$\begin{aligned}P(x_1,t_1) &= \tfrac{1}{2}\exp(-\lambda t_1)\delta(x_1-x_0-at_1) + \tfrac{1}{2}\exp(-\lambda t_1)\delta(x_1-x_0+at_1)\\ &\quad+\frac{\lambda}{2a}\exp(-\lambda t_1)\Theta\left(t_1+\frac{x_1-x_0}{a}\right)\Theta\left(t_1-\frac{x_1-x_0}{a}\right)\\ &\quad\times\left\{I_0\left(\lambda\sqrt{t_1^2-\left(\frac{x_1-x_0}{a}\right)^2}\right)\right.\\ &\quad\left.+\frac{t_1}{\sqrt{t_1^2-\left(\frac{x_1-x_0}{a}\right)^2}}I_1\left(\lambda\sqrt{t_1^2-\left(\frac{x_1-x_0}{a}\right)^2}\right)\right\}.\end{aligned} \tag{B7}$$

Here $I_0(z)$ *and* $I_1(z)$ *are the modified Bessel functions [12] and* x_0 *is the initial condition. This expression is valid for a stationary dichotomous noise, i.e. when its correlation function is characterized by* $\langle\xi(t)\xi(t')\rangle = a^2\exp(-2\lambda|t-t'|)$, *where a is the amplitude of the dichotomous noise. The two travelling Dirac deltas appearing in the probability distribution come from the* *deterministic realizations of $\xi(t)$, i.e. solutions of $\dot{X}_0(t) = \pm a$. Of course these Dirac deltas are attenuated in time by an exponential factor controlled for the parameter λ (the hopping rate of the dichotomous noise).*

Example 3. *A* Poisson noise $\xi(t)$ *(i.e. a Campbell's white shot noise [1, 2, 6, 15]) has the characteristic functional* $G_\xi([k(t)]) = \exp \rho \int_0^\infty (\exp(\mathrm{i}Ak(t)) - 1)\,\mathrm{d}t$*; where ρ represents the average number of events (pulses) per unit of time, and A is the amplitude of the Dirac pulses. In this case, the SP $\boldsymbol{X}_0(t)$ gives a simple jump process (jumps of amplitude A from the initial condition) with a Poisson distribution in the time axis. Its characteristic functional is given by*

$$G_{X_0}([Z(t)]) = \mathrm{e}^{+\mathrm{i}k_0 X_0(0)} \exp \rho \int_0^\infty \left(\exp\left(\mathrm{i}A \int_t^\infty Z(s)\,\mathrm{d}s \right) - 1 \right) \mathrm{d}t \tag{B8}$$

from which follows

$$G_{X_0}(k_1, t_1) = \exp[\rho t_1(\mathrm{e}^{\mathrm{i}Ak_1} - 1) + \mathrm{i}k_1 X_0(0)]. \tag{B9}$$

Example 4. *A* radioactive noise $\xi(t)$ *has the characteristic functional [6, 16]* $G_\xi([k(t)]) = [\beta \int_0^\infty \exp(-t\beta + \mathrm{i} \int_0^t k(s)\,\mathrm{d}s)\,\mathrm{d}t]^{\xi_0}$*. Here ξ_0 represents the number of active nucleus at $t = 0$, and β is the probability per unit of time that an individual decay occurs. This particular noise is not stationary and at long-time the noise is always null. Using this radioactive noise, from (B1) the characteristic functional of the unbounded SP $\boldsymbol{X}_0(t)$, reads*

$$G_{X_0}([Z(t)]) = \left[\beta \int_0^\infty \exp\left(-t\beta + \mathrm{i} \int_0^t \mathrm{d}s \int_s^\infty Z(u)\,\mathrm{d}u \right) \mathrm{d}t \right]^{\xi_0} \mathrm{e}^{+\mathrm{i}k_0 X_0(0)}. \tag{B10}$$

Therefore we get for the characteristic function

$$G_{X_0}(k_1, t_1) = \left[\beta \frac{\exp[(\mathrm{i}k_1 - \beta)t_1] - 1}{\mathrm{i}k_1 - \beta} + \exp[(\mathrm{i}k_1 - \beta)t_1] \right]^{\xi_0} \exp \mathrm{i}k_1 X_0(0). \tag{B11}$$

Using Newton's binomial, the 1-time probability distribution $P(x_1, t_1)$ is given by†

$$P(x_1, t_1) = \exp(-\beta\xi_0 t_1)\delta(x_1 - X_0(0) - \xi_0 t_1) + \exp[-\beta(x_1 - X_0(0)\ )] \times \sum_{n=1}^{\xi_0} \beta^n \binom{\xi_0}{n} \sum_{m=0}^{n} \frac{(-1)^{m+n}}{\Gamma(n)} \binom{n}{m} [x_1 - X_0(0) - (\xi_0 - m)t_1]^{n-1}. \tag{B12}$$

As in the dichotomous noise case, here the travelling Dirac delta—in the probability distribution—comes from the deterministic realization of the noise: $\xi(t) = \xi_0$ (i.e. the solution of $\dot{X}_0(t) = \xi_0$). Now, this Dirac delta is attenuated in the time by an exponential factor controlled by the product $\beta\xi_0$. This is because the probability to lose the deterministic realization is proportional to the occurrence of an individual decay, and there are ξ_0 different ways that can occur.

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† Here we have used that $\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k_1 \frac{\exp(-\mathrm{i}k_1 x)}{(\mathrm{i}k_1 - \beta)^n} = \frac{(-1)^n}{\Gamma(n)} x^{n-1} \exp(-\beta x)$.

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